DECONVOLUTION WITH GAUSSIAN BLUR PARAMETER AND HYPERPARAMETERS ESTIMATION

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ABSTRACT

This paper proposes a Bayesian approach for unsupervised image deconvolution when the parameter of the gaussian PSF is unknown. The parameters of the regularization parameters are also unknown and jointly estimated with the other parameters. The solution is found by inferring on a global *a posteriori* law for unknown object and parameters. The estimate is chosen in the sense of the posterior mean, numerically calculated by means of a Monte-Carlo Markov chain algorithm. The computation is efficiently done in Fourier space and the practicability of the method is shown on simulated examples. Results show high-frequencies restoration in the estimated image with correct estimation of the hyperparameters and instrument parameters.

Index Terms— Image restoration, unsupervised deconvolution, myopic deconvolution, full-bayesian approach, Monte-Carlo Markov chain.

1. INTRODUCTION

Deconvolution is an active research field [1, 2]. Examples of application are medical imaging, astronomy, nondestructive testing and more generally imagery problems. The deconvolution problem is ill-posed and a well-known solution relies on the introduction of prior information in addition to the data. The resulting estimate depends on two sets of variables in addition to the data. Firstly, the solution depends on the parameters of the probability laws named hyperparameters (mean, variance, parameters of correlation matrix,. . .). Secondly, the estimate naturally depends on the instrument response model. These problems are called myopic or blind and a book has been recently published on this subject [1]. Many application have knownledge about the shape of the PSF up to some unknown parameters. An exemple where the instrument is known up to some shape parameters is microscopy imaging [4]. This paper deal with the case where the PSF is known to be gaussian but the width parameter is unknwon. This model is classical in optical imaging for exemple where the *Airy* disc is approximated by a gaussian [3]. This model is also encoutered in astmospheric turbulence, focus or defocus modeling. A difficulty is the non-linearity of the likelihood of the width parameter.

A recent paper [5] address the estimation of the gaussian blur with an empirical method. Several value are used to estimate the image with a Wiener filter with fixed hyperparameter. The best parameter is chosen to minimize the second derivatives L_1 norm of the estimate. Another work [6] propose a maximum likelihood to estimate the blur parameter. Our approach is to estimate the parameter jointly with the image and the hyperparameter, not in two step. The paper [2] use a Gibbs sampler to estimate jointly the image and all the pixel of the PSF. This approach estimates the whole PSF but is not adapted to estimate the parameter of a gaussian blur since they doesn't take into account the knownledge about the PSF shape. In addition the complexity of this model make the computational cost very high.

We propose a new method that, contrary to [5] or [6], jointly estimates the image, the hyperparameters and the gaussian blur parameter in common framwork. The estimate is chosen as the mean of the posterior law and is computed using MCMC algorithms, as in [2], to obtain sample of the *a posteriori* law despite of its complexity. The model allows the computation to be done in Fourier space in a very effective manner.

2. FORWARD MODEL

We consider N pixels real square images represented in lexicographic order by vector $\boldsymbol{x} \in \mathbb{R}^N$, with generic elements x_n. The forward model is written $\boldsymbol{y} = \boldsymbol{H_w x} + \boldsymbol{n}$ where $\boldsymbol{y} \in \mathbb{R}^N$ are the data, $\boldsymbol{H_w}$ a convolution matrix parametrized by parameters collected in $\boldsymbol{w}$ and $\boldsymbol{n}$ the model errors. In these paper we deal with a gaussian blur written in Fourier space

$$\mathring{h}(\nu_\alpha, \nu_\beta) = \exp\Big(-2\pi^2 w\big(\nu_\alpha^2 + \nu_\beta^2\big)\Big) \quad (1)$$

with frequencies $(\nu_\alpha, \nu_\beta) \in [-0.5;\, 0.5]^2$. The non-linear dependency of the value h with w will be the main difficulty. The matrix $\boldsymbol{H_w}$ is considered block-circulant circulant-block (BCCB) for computational efficiency of the convolution in Fourier space. The diagonalization of $\boldsymbol{H_w}$ is written $\boldsymbol{\Lambda_H} = \boldsymbol{F}\boldsymbol{H_w}\boldsymbol{F}^\dagger$ where $\boldsymbol{\Lambda_H}$ is a diagonal matrix, $\boldsymbol{F}$ the unitary Fourier matrix and † the transpose conjugate symbol. The convolution, in Fourier space, is written $\mathring{\boldsymbol{y}} = \boldsymbol{\Lambda_H}\mathring{\boldsymbol{x}} + \mathring{\boldsymbol{n}}$ where $\mathring{\boldsymbol{x}} = \boldsymbol{F}\boldsymbol{x}$, $\mathring{\boldsymbol{y}} = \boldsymbol{F}\boldsymbol{y}$ and $\mathring{\boldsymbol{n}} = \boldsymbol{F}\boldsymbol{n}$ are the 2D discrete Fourier transform (DFT-2D) of image, data and noise, respectively. The description is equivalent and everything will be done in Fourier space.

3. BAYESIAN FRAMEWORK

This section presents the prior law for each set of parameters. In order to account for smoothness, the image law introduces penalization of high-frequency through a difference operator on the pixel. Conjugate law for the hyperparameters and uniform law for the instrument parameters are considered.

3.1. Image prior law

The probability law for the image is a toroidal Gaussian field $p(\boldsymbol{x}|\gamma_x) \sim \mathcal{N}(0, (\gamma_x \boldsymbol{D}^\dagger \boldsymbol{D})^{-1})$ parametrized by the γ_x precision. In the Fourier space, the probability law is also Gaussian and writes

$$p(\mathring{\boldsymbol{x}}|\gamma_x) \propto \gamma_x^{(N-1)/2} \exp\left[-\frac{\gamma_x}{2}||\boldsymbol{\Lambda_D}\mathring{\boldsymbol{x}}||^2\right]. \qquad (2)$$

The circulant difference operator $\boldsymbol{D}$, and its diagonalization $\boldsymbol{\Lambda_D} = \boldsymbol{F}\boldsymbol{D}\boldsymbol{F}^\dagger$, is build with a high-pass filter, the Laplacian for example.

3.2. Noise and data laws

The noise is modeled as white zero-mean Gaussian with unknown precision parameter γ_n. Consequently the likelihood of the parameters given the data writes

$$p(\mathring{\boldsymbol{y}}|\mathring{\boldsymbol{x}}, \gamma_n, \boldsymbol{w}) \propto \gamma_n^{N/2} \exp\left[-\frac{\gamma_n}{2}||\mathring{\boldsymbol{y}} - \boldsymbol{\Lambda_H}\mathring{\boldsymbol{x}}||^2\right] \qquad (3)$$

It depends, of course, on the image $\mathring{\boldsymbol{x}}$, on the noise parameter γ_n and instrument parameters $\boldsymbol{w}$ embedded in $\boldsymbol{\Lambda_H}$.

3.3. Hyperparameters law

A classical choice for hyperparameter prior law is conjugate law with computational efficiency justification [2]. A conjugate law for Gaussian precisions parameters is the Gamma law parametrized by two values (α_i, β_i), with $i = x$ or n,

$$p(\gamma_i) = \frac{1}{\beta_i^{\alpha_i}\Gamma(\alpha_i)}\gamma^{\alpha_i - 1} \exp\left(-\gamma_i/\beta_i\right). \qquad (4)$$

In addition we also want to use non-informative *a priori* law. With specific parameter values, one obtains the non-informative Jeffreys's prior law $p(\boldsymbol{\gamma}) = 1/\boldsymbol{\gamma}$ with $(0, +\infty)$.

3.4. Gaussian blur parameter law

For the blur parameter w, we consider that a physical study provides a nominal value with uncertainty in a given interval $[m\ M]$. These is the case for example in optics where FHWM of the *Airy* disc is the wavelength over the lens diameter λ/D [3]. Since no more information is available, we consider a uniform prior on the interval

$$p(w) = \mathcal{U}_{[m\ M]}(w) = \frac{1}{M - m}\mathbb{1}_{[m\ M]}(w) \qquad (5)$$

with $\mathbb{1}_{[m\ M]}(w) = 1$, if $w \in [m\ M]$, 0 elsewhere. Other choice are possible but do allow easier computation because of the non-linear dependency in the likelihood.

3.5. Posterior law

At this point the law of the image, the hyperparameters, the instrument parameters and the data are available. Thus, the *a posteriori* law for all the parameters is built by multiplying the likelihood (3) and the *a priori* laws (2), (4) and (5)

$$p(\mathring{\boldsymbol{x}}, \gamma_n, \gamma_x, w|\mathring{\boldsymbol{y}}) \propto \gamma_n^{N/2-1}\gamma_x^{(N-1)/2-1}\mathbb{1}_{[m\ M]}(w) \\ \exp\left[-\frac{\gamma_n}{2}||\mathring{\boldsymbol{y}} - \boldsymbol{\Lambda_H}\mathring{\boldsymbol{x}}||^2 - \frac{\gamma_x}{2}||\boldsymbol{\Lambda_D}\mathring{\boldsymbol{x}}||^2\right]. \qquad (6)$$

Finally, inference is done on this law (6). An estimate and the algorithm is described in the next section.

4. POSTERIOR MEAN AND LAW EXPLORATION

To compute the posterior mean of the parameters, Monte Carlo Markov chain is used to provides samples of (6). The samples are obtained by a Gibbs sampling algorithm. It consists in sampling, iteratively, a conditional posterior law of a set of parameters given all the others parameters obtained at previous iteration.

4.1. Sampling the image

The conditional posterior law of the image is a Gaussian law. Its covariance matrix is diagonal and writes

$$\boldsymbol{\Sigma}^{(k+1)} = \gamma_n^{(k)}|\boldsymbol{\Lambda}_{\boldsymbol{H}}^{(k)}|^2 + \gamma_x^{(k)}|\boldsymbol{\Lambda_D}|^2 \qquad (7)$$

and the mean

$$\boldsymbol{\mu}^{(k+1)} = \gamma_n^{(k)}\left(\boldsymbol{\Sigma}^{(k+1)}\right)^{-1}\boldsymbol{\Lambda}_{\boldsymbol{H}}^{*\,(k)}\mathring{\boldsymbol{y}}. \qquad (8)$$

where $*$ is the conjugate symbol. The vector $\boldsymbol{\mu}^{(k+1)}$ is the regularized least square solution at current iteration (or the Wiener-Hunt solution). Finally, since the matrix are diagonal, the sampling of the image is very effective: all the operation are term-wise addition and multiplication.

4.2. Sampling precision parameters

The conditional posterior laws of the precisions are Gamma. For γ_n and γ_x the parameters law are

$$\alpha_n^{(k+1)} = N/2, \qquad \beta_n^{(k+1)} = 2/||\mathring{\boldsymbol{y}} - \boldsymbol{\Lambda}_{\boldsymbol{H}}^{(k)} \mathring{\boldsymbol{x}}^{(k+1)}||^2,$$
$$\alpha_x^{(k+1)} = (N-1)/2, \quad \beta_x^{(k+1)} = 2/||\boldsymbol{\Lambda}_{\boldsymbol{D}} \mathring{\boldsymbol{x}}^{(k+1)}||^2.$$

4.3. Sample instrument parameters

The conditional law for instrument parameters writes

$$w^{(k+1)} \propto \exp\left[-\frac{\gamma_n^{(k+1)}}{2} ||\mathring{\boldsymbol{y}} - \boldsymbol{\Lambda}_{\boldsymbol{H},w}\, \mathring{\boldsymbol{x}}^{(k+1)}||^2 \right]. \qquad (9)$$

This law is not standard and intricate, and no algorithm exists for direct sampling. In addition the dependency of $\boldsymbol{\Lambda}_{\boldsymbol{H},w}$ with w is non-linear. The proposed solution relies on the powerful Metropolis-Hastings method. In the independent form the algorithm is:

1. Sample a proposition $w_\mathrm{p} \sim p(w) = \mathcal{U}_{[m\ M]}(w)$.
2. Calculate the criterion
$$J(w^{(k)}, w_\mathrm{p}) = \frac{\gamma_n^{(k+1)}}{2} \Big(||\mathring{\boldsymbol{y}} - \boldsymbol{\Lambda}_{\boldsymbol{H},w^{(k)}}\, \mathring{\boldsymbol{x}}^{(k+1)}||^2 - ||\mathring{\boldsymbol{y}} - \boldsymbol{\Lambda}_{\boldsymbol{H},w_\mathrm{p}}\, \mathring{\boldsymbol{x}}^{(k+1)}||^2 \Big).$$
3. Sample $t \sim \mathcal{U}_{[0\ 1]}$ and takes $w^{(k+1)} = w_\mathrm{p}$ if $\log t < \min\{J, 0\}$, $w^{(k+1)} = w^{(k)}$ otherwise.

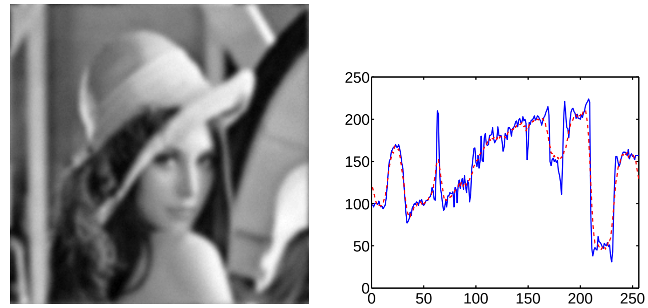
Since everything is in Fourier space, and w is a scalar, the algorithm is very effective. As a counter part, more sample are needed because of rejection.

4.4. Empirical mean

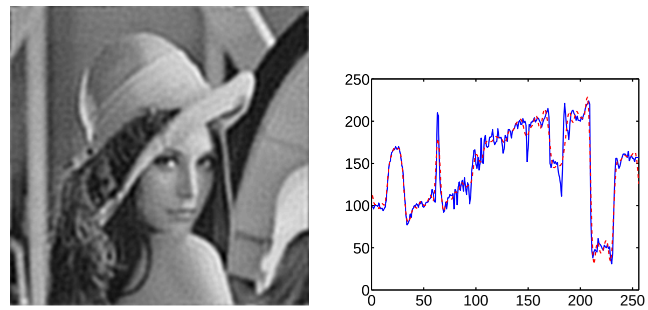
The sampling of $\mathring{\boldsymbol{x}}$, $\boldsymbol{\gamma}$ and w are repeated iteratively until the law has been sufficiently explored. The estimate is approximated with $\hat{\boldsymbol{x}} = \boldsymbol{F}^\dagger \mathbb{E}[\mathring{\boldsymbol{x}}]$ where all the iteration can be done in Fourier space with an unique IFFT at the end.

5. DECONVOLUTION RESULTS

This section is devoted to numerical experiments. It is based on two images : (1) the usual Lena case and a (2) the case of a sample of the prior law (so that true values of the hyperparemeters γ_n and γ_x are known). The noise is a white gaussian and several values of γ_n are tested. The matrix $\boldsymbol{\Lambda}_{\boldsymbol{D}}$ is obtained with the FFT-2D of the Laplacian $[0\ 1\ 0; 1-4\ 1; 0\ 1\ 0]\,/8$. The width parameter w is set to 4 or 6. It's *a priori* laws is $p(w) = \mathcal{U}_{[2\ 7]}$. This corresponds to uncertainty of approximately $\pm$ 40% around the nominal value.



(a) Data



(b) Estimate

Fig. 1. Result for Lena. Fig. 1(a) is the data. Fig. 1(b) is the estimate. Profiles correspond to the 68-th line. Solid line profile is the true.

5.1. Estimation results

The result for the image is illustrated Fig. 1(b). The image is restored, more details are visible and the profiles are closer to the true image than data. High-frequencies are more visible and oscillation that were not visible in the data are present in the estimate particularly around pixels 200 in the profile Fig. 1(b). The estimated circular mean of the power spectral density of the objects are illustrated Fig. 2. The spectrum of the true image is retrieve up to the frequency $f \approx 0.15$ limits where the noise start to be dominant. After this frequency, the power spectral density of the data mainly comes from the noise. Since the method estimate the parameters γ_n and γ_x, this frequency limit is automatically estimated.

Concerning the hyperparameters, since we must know the true value of all the parameter, the study is done on a sample of the prior law. Their estimates are reported in Tab. 1. For each result the ISNR define as $10\log_{10}(||\boldsymbol{x}-\boldsymbol{y}||^2/||\boldsymbol{x}-\hat{\boldsymbol{x}}||^2)$ increase. The estimated of γ_n is each time close to the true values and seems to be a little under estimated. The γ_n estimation is very close with $\widehat{\gamma}_n = 0.49$ instead of 0.5 or 1.98 for 2. The value of γ_x is underestimated with approximately 1.9 and 1.3 instead of 2.

For the gaussian blur the result is compared to the method DL_1 described in [5] that also estimate the gaussian blur parameter. Our result is each time closer to the true value of the paramater specially when the noise increases. The incertitude

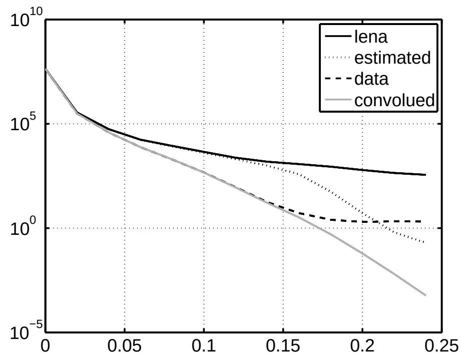


Fig. 2. Circular mean of the power spectral density (PSD) of the image, the output model image $\boldsymbol{Hx}$, the data (filtered image corrupted by noise) and the estimate that is close to the true until noise dominate in data.

	γ_n (SNR)	γ_x	w	ISNR
True	**0.5** (27)	**2**	**6**	
mean $\pm\sigma$	0.49 ± 0.01	2.06 ± 0.14	6.03 ± 0.3	3.85
DL_1	-	-	9.23	
True	**0.01** (10)	**2**	**6**	
mean $\pm\sigma$	0.009 ± 0.002	2.05 ± 0.2	5.35 ± 0.8	9.09
DL_1	-	-	6.5	
True	**0.5** (32)	**2**	**4**	
mean $\pm\sigma$	0.49 ± 0.01	1.3 ± 0.09	4.4 ± 0.29	2.86
DL_1	-	-	2.05	

Table 1. Paramters results. SNR and ISNR are in dB.

is quite small. Even with the prior incertitude the posterior law is concentrate around the true value.

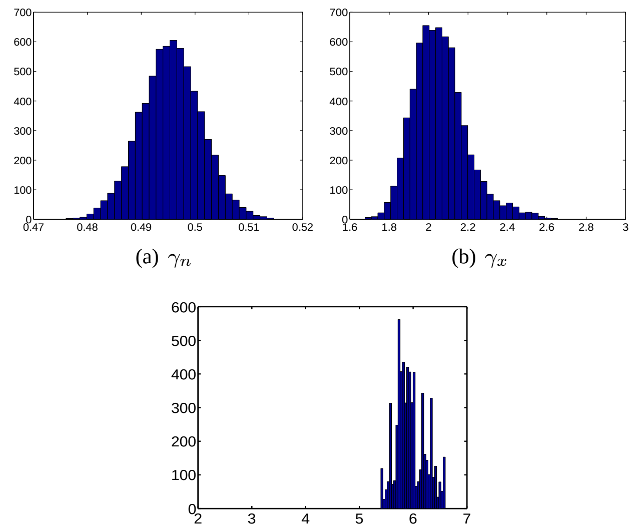
5.2. *A posteriori* law characteristics

The histograms of γ_n and γ_x, Fig. 3(a) and 3(b) respectively, are concentrated around their mean. The variance of γ_n is lower than the γ_x one. Effectively there is a degradation, by the convolution, of the information about γ_x present in the image which is an input of the instrument model. This is not the case for γ_n, which is directly observed in the output, resulting in a lower variance for γ_n.

The histogram of the instrument parameter is different. The histogram of w Fig. 3(c) is much more concentrated around the true value than the hyperparameter histograms. It's variances is quite small with regards to the interval of the *a priori* law.

6. CONCLUSION

This paper presents a new global and coherent method for the estimation of a gaussian blur parameter in unsupervised deconvolution. It is build within a Bayesian framework and a extended *a posteriori* law for the image, the hyperparameters and the blur parameter. The estimate, defined as the posterior mean, is computed by means of an MCMC algorithm in less than 5 minutes on a standard computer. The results show that the deconvolved image is closer to the true image than the data and show restored high-frequencies. In addition the gaussian blur parameter and the hyperparameters are estimated and close to the true value.



(a) γ_n (b) γ_x

(c) w

Fig. 3. Histograms for γ_n and γ_x and w.

7. REFERENCES

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